stable regardless of the initial points $u_i(0)$ or $\mathbf{x}(0)$. Repeating the same argument for all flows i, we establish their individual convergence.

The necessity of condition $0 < \beta < 2$ directly follows from Corollary 3.

Combining EMKC's Lyapunov and global quasi-asymptotic stability, we have:

Corollary 4: EMKC is globally asymptotically stable under constant feedback delay D if and only if $0 < \beta < 2$.

VI. Conclusion

This paper offered a comprehensive stability analysis of a new congestion controller called MKC, which is proven to be locally stable with arbitrary (heterogeneous) feedback delays under easily verifiable conditions. This property makes MKC a highly appealing platform for congestion control in future high-speed networks with heterogeneous users. Moreover, we proposed a *negative* packet-loss feedback function to be used in conjunction with MKC and called the resulting controller EMKC. We proved that EMKC achieves both RTT-independent stability and fairness and converges to link utilization exponentially fast. Our investigation of global stability shows that all EMKC flows converge to their unique stationary points regardless of the initial point in which the system is started. We proved this fact for constant delays D and our future work is to extend the analysis to heterogenous delays.

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Controllability and Observability for a Class of Controlled Switching Impulsive Systems

Bin Liu and Horacio J. Marquez

***Abstract*—In this note, we study the controllability and observability problem for a class of controlled switching impulsive systems. By proposing several formula of variation of parameters for this type of time-varying systems and employing the characteristic polynomial theory of matrix, we establish necessary and sufficient conditions for controllability and controlled observability with respect to a given switching time sequence. Specializing the obtained results to the case of time-invariant linear switching impulsive systems, we derive some simple algebraic criteria, which include the results reported in the literature for time-invariant linear switching systems, linear impulsive systems and classical linear systems. One example is worked out for illustration.**

***Index Terms*—Controllability, controlled observability, hybrid systems, switching impulsive systems, variation of parameters.**

I. Introduction

Motivated by the fact that hybrid systems provide a natural framework for mathematical modeling of many physical phenomena, their study has received considerable attention for the last two decades [1]–[3]. Most of the work encountered in the literature has focussed on two types of hybrid systems, namely; switching and impulsive systems. See [4]–[10] and the references cited therein for recent work on these two classes of systems. It is, however, worth noticing that switching and impulsive systems do not include some important hybrid systems existing in some applications characterized by switches of the states and abrupt changes at the switching instants.

Indeed, in many natural phenomena in systems such as evolutionary processes, biological neural networks and bursting rhythmic models in pathology, when certain quantities accumulate, the nature of the reaction undergoes an abrupt change. In this case, one needs to switch to a new system of differential equations taking into consideration a momentary perturbation of an impulsive nature. This class of systems exhibit simultaneously continuous-time dynamic switching and impulsive jump phenomena. A general description of these systems is called switching impulsive system. Examples include evolutionary processes and biological systems, as well as frequency-modulated signal processing systems, networked control systems, optimal control models in economics, and flying object motions. See [22]–[28].

The controllability and observability problem for hybrid systems has recently received considerable attention. See [11]–[18], [20], [21]. In [11], controllability and observability of periodic switching linear systems was studied. In [18] and [31], geometric criteria for controllability of switching systems was established. In [30], controllability of

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B. Liu is with the Department of Information and Computation Sciences, Hunan University of Technology, Zhuzhou 412008, China, and also with the Department of Information Engineering, The Australian National University, Canberra ACT 0200, Australia. (e-mail: bin.liu@anu.edu.au).

H. J. Marquez is with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB T6G 2V4 Canada. (e-mail: marquez@ece.ualberta.ca).

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switching bilinear systems was investigated using Lie algebraic techniques. In [12], [15] and [21], the controllability problem was studied for linear impulsive hybrid systems. Most of the literature on the subject, however, deals with time-invariant systems almost exclusively. Moreover, so far very few results for time-variant switching impulsive systems in which both switching and impulse are simultaneously considered have been reported.

In this note, we study the controllability and observability problem for a class of time-variant controlled switching impulsive systems, where the impulses occur at the switching instants. We derive controllability and controlled observability criteria expressed in the form of a matrix rank condition that is much easier to check than the geometric criteria previously reported for time-invariant switched systems ([17], [18]). Moreover, when specializing to the case of time-invariant linear hybrid systems, our results coincide with those previously reported in the literature (see e.g., [11], [12], [15], [18], [20], [21]) for time-invariant linear switching systems, linear impulsive systems and classical linear systems. The extension considered here is, however, nontrivial, as we deal with a much broader class of systems.

The rest of this note is organized as follows. In Section II, we present preliminaries and propose the results of variation of parameters for the system. In Sections III–IV, we establish necessary and sufficient conditions for controllability and controlled observability, respectively. For illustration, one representative example is given in Section V. In Section VI we provide our conclusions and final remarks.

II. Preliminaries

In the sequel, R^n denotes the n-dimensional Euclidean space and $R_+ = [0, +\infty)$. For any positive integers j, k $(j \le k)$, denote $\prod_{i=k}^{j} a_i = a_k a_{k-1} \cdots a_j$, where $a_i \in R$. Let I denote the $n \times n$ identity matrix.

Consider the time-variant controlled switching impulsive system of the form

$$\begin{aligned} \dot{x}(t) &= A_{q(t)}(t)x(t) + B_{q(t)}(t)u_{q(t)}(t), \ \ t_k < t \le t_{k+1} \\ \triangle x &= C_{q(t)}(t)x(t) + D_{q(t)}(t)v_{q(t)}(t), \ t = t_k \\ y(t) &= F_{q(t)}(t)x(t) \end{aligned} \tag{1}$$

where $A_{q(t)}(t) \in R^{n\times n}$, $B_{q(t)}(t) \in R^{n\times m}$, $C_{q(t)}(t) \in R^{n\times n}$, $D_{q(t)}(t) \in R^{n\times l}$, $F_{q(t)}(t) \in R^{p\times n}$, with all the entries in these matrices being continuous scalar functions of $t \in R_+$; $0 \le t_0 < t_1 < \cdots < t_M < t_{M+1} = t_f$ is the switching time sequence, at which the impulses occur, $x(t) \in R^n$ is the state, $y(t) \in R^p$ is the control output, $u_{q(t)}(t) \in U_c \subset R^m$ is a continuous input, $v_{q(t)}(t) \in U_d \subset R^l$ is a discrete input signal and $q : [t_0, t_f] \to Q = \{1, 2, \ldots, r\}$ is the switching control signal of the system. Since Q is finite, the switching control $q(t)$ is necessarily piecewise constant with values in Q. We normalize the switching control $q(t)$ to be left continuous on the time interval $[t_0, t_f]$, i.e., $q(t) = q(t^-) = \lim_{s\to t^-} q(s)$. Let $x(t)$ denote the solution at time t of system (1) starting from $x(t_0) = x_0$.

Remark 2.1:

1) In engineering applications, the cost of controllers is often important. To reduce unnecessary costs, the controllers $u_{q(t)}(t)$ and $v_{q(t)}(t)$ usually accommodate to one of the following two cases: (i) $D_{q(t)}(t) = 0$, i.e., the controllers are added only to the continuous part; or (ii) $B_{q(t)}(t) = 0$, i.e., the controllers are added only to the discrete part.
2) In system (1), the impulses $\triangle x$ and the mode changes q occur at the same time. Otherwise, if there exists some switching time t_k at which impulse does not occur, then, we set $\triangle x = 0$ at time t_k. On the other hand, if there is some impulsive instance t_k at which the mode change does not occur, i.e., $q(t_k) = q(t_k^+)$, then, we look this case as there is a mode change between the same subsystem. Hence, for convenience, we put the impulse instances and switching times as the same.

Definition 2.1: For the given switching time sequence $\Sigma = \{t_i\}_{i=0}^{M+1}$ with $t_{M+1} = t_f$, function $q : [t_0, t_f] \to Q$ is said to be an admissible switching control if

i) $q(t) \equiv q(t_0) = q(t_1)$, for all $t \in [t_0, t_1]$;
ii) $q(t) \equiv q(t_i^+) = q(t_{i+1})$, for $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, M$.

Remark 2.2: It should be noticed that for any switching system, the switching control $q(t)$ includes two parts (see e.g., [23], [27]): (i) when should the switching control be applied? i.e., $t_k = ?$ $(k = 1, 2, \ldots,)$; and (ii) which system should be activated at the switching times t_k, $(k = 1, 2, \ldots)$,? i.e., $q(t_k^+) = ?$ $(k = 1, 2, \ldots)$. Hence, if the switching time sequence $\Sigma = \{t_i\}_{i=0}^{M+1}$ with $t_{M+1} = t_f$ is given, then there remains to consider the second part of $q(t)$. In this note, for the given switching time sequence $\Sigma = \{t_i\}_{i=0}^{M+1}$ with $t_{M+1} = t_f$, we consider an admissible switching control.

Corresponding to system (1), for every $q(t) \in Q$, consider linear time-varying system:

$$\dot{x}(t) = A_{q(t)}(t)x(t). \tag{2}$$

Suppose that $X_{q(t)}(t)$ is the fundamental solution matrix of (2). Then, for any switching-times $\{t_i\}_{i=0}^{M}$, and $t, s \in (t_i, t_{i+1}], i = 0, 1, \ldots, M$, the transition matrix associated with $A_{q(t)}(t)$ is $X_{q(t_{i+1})}(t, s) = X_{q(t_{i+1})}(t)X^{-1}_{q(t_{i+1})}(s)$. It is easy to see that, for any $t, s, \tau \in R_+$, and $t, s, \tau \in (t_i, t_{i+1}]$, $X_{q(t_{i+1})}(t, \tau)X_{q(t_{i+1})}(\tau, s) = X_{q(t_{i+1})}(t, s)$, $X_{q(t_{i+1})}(t, t) = I$, where I is the identity matrix of order n, $X_{q(t_{i+1})}(t, s) = X^{-1}_{q(t_{i+1})}(s, t)$ (see [19]).

Lemma 2.1: Let $I_{q(t)}(t) = C_{q(t)}(t)x(t) + D_{q(t)}(t)v_{q(t)}(t)$. For the switching time sequence $\Sigma = \{t_i\}_{i=0}^{M+1}$, and $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, M$, the solution of the system (1) satisfies (3), shown at the bottom of the page.

Proof: By using variation of parameters of ordinary differential system [19] and an induction argument, we can derive the result of this lemma. The details are omitted. ■

Lemma 2.2: In system (1), assume that $D_{q(t)}(t) = 0$, and $C_{q(t_0)}(t_0) = 0$, then, for the switching time sequence $\Sigma = \{t_i\}_{i=0}^{M+1}$, and $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, M$, the solution of system (1) satisfies (4), shown at the bottom of the next page.

Proof: The proof follows an argument analogous to that of Lemma 2.1 and it is omitted. ■

Lemma 2.3: For any matrix $X \in R^{n\times n}$, there exist linearly independent functions $f_i(t), i = 0, 1, \ldots, n-1$, such that the following equation holds:

$$e^{Xt} = \sum_{i=0}^{n-1} f_i(t)X^i. \tag{5}$$

$$\begin{aligned} x(t) = X^{-1}_{q(t_{k+1})}(t_{k+1}, t)\Bigg\{ &\prod_{i=k+1}^{1} X_{q(t_i)}(t_i, t_{i-1})x_0 + \sum_{j=1}^{k}\prod_{i=k+1}^{j+1} X_{q(t_i)}(t_i, t_{i-1})\int_{t_{j-1}}^{t_j} X_{q(t_j)}(t_j, s)B_{q(t_j)}(s)\cdot u_{q(t_j)}(s)ds \\ &+ \sum_{j=1}^{k}\prod_{i=k+1}^{j+1} X_{q(t_i)}(t_i, t_{i-1})I_{q(t_j)}(t_j) + \int_{t_k}^{t} X_{q(t_{k+1})}(t_{k+1}, s)B_{q(t_{k+1})}(s)u_{q(t_{k+1})}(s)ds\Bigg\}. \end{aligned} \tag{3}$$

Proof: A direct consequence of the Cayley–Hamilton theorem and Lemma 2 in [12]. ■

III. Controllability

In this section, for a given switching time sequence (i.e., impulse instance sequence) $\Sigma = \{t_i\}_{i=1}^{M+1}$, we introduce the concept of controllability of system (1) and investigate the conditions under which the system is controllable.

Definition 3.1: The switching impulsive system (1) is said to be controllable on $[t_0, t_f]$ $(t_f > t_0)$ with respect to the switching time sequence $\Sigma = \{t_i\}_{i=1}^{M+1}$ with $t_{M+1} = t_f$ if for all $x_0, x_f \in R^n$ there exist an admissible switching control input $q(t)$ and continuous control inputs $u_{q(t)}(t)$ (or discrete control inputs $v_{q(t)}(t)$) such that (1) has a solution $x(t)$ existing on $[t_0, t_f]$ satisfying $x(t_0) = x_0$ and $x(t_f) = x_f$.

Definition 3.2: The switching impulsive system (1) is said to be controllable to the origin on $[t_0, t_f]$ $(t_f > t_0)$ with respect to the switching time sequence Σ with $t_{M+1} = t_f$ if for all $x_0 \in R^n$ there exist an admissible switching control input $q(t)$ and continuous control inputs $u_{q(t)}(t)$ (or discrete control inputs $v_{q(t)}(t)$) such that (1) has a solution $x(t)$ existing on $[t_0, t_f]$ satisfying $x(t_0) = x_0$ and $x(t_f) = 0$.

Remark 3.1: It was shown in [14] that controllability of an impulsive systems is not equivalent to controllability to the origin.

A. Only Continuous Control Signals are Input: $D_{q(t)}(t) = 0$

Without loss of generality, let $C_{q(t_0)}(t_0) = 0$. Denote $E_{M+1} = I$, and for $j = 0, 1, \ldots, M$

$$E_j = \prod_{i=M+1}^{j+1} X_{q(t_i)}(t_i, t_{i-1})(I + C_{q(t_{i-1})}(t_{i-1})). \tag{6}$$

Then, it follows from Lemma 2.2 that

$$x(t_f) = E_0 x_0 + \sum_{j=1}^{M+1} E_j \int_{t_{j-1}}^{t_j} X_{q(t_j)}(t_j, s) B_{q(t_j)}(s) u_{q(t_j)}(s) ds. \tag{7}$$

Theorem 3.1: Let $D_{q(t)}(t) = 0$, $C_{q(t_0)}(t_0) = 0$ and assume that every matrix $I + C_{q(t_j)}(t_j)$ is non-singular. Then, the switching impulsive system (1) is controllable on $[t_0, t_f]$ with respect to the switching time sequence Σ if and only if there exists an admissible switching control input $q(t)$ such that

$$rank(\Phi) = n \tag{8}$$

where $\Phi = (\Phi_1\ \Phi_2\ \cdots \Phi_{M+1})$, and for $j = 1, 2, \ldots, M+1$

$$\Phi_j = \int_{t_{j-1}}^{t_j} E_j X_{q(t_j)}(t_j, s) B_{q(t_j)}(s) \cdot B_{q(t_j)}^T(s) X_{q(t_j)}^T(t_j, s) E_j^T ds.$$

Proof: Denote $\tilde{\Phi} = \sum_{j=1}^{M+1} \Phi_j$. It is easy to see that $rank(\Phi) = rank(\tilde{\Phi})$.

Sufficiency: If there exists an admissible switching control input $q(t)$ such that $rank(\Phi) = n$, we show system (1) is controllable on $[t_0, t_f]$. By Definition 3.1 and Lemma 2.2 and (7), we only need to prove that there exist control inputs $u_{q(t)}(t)$ such that

$$x_f - E_0 x_0 = \sum_{j=1}^{M+1} E_j \int_{t_{j-1}}^{t_j} X_{q(t_j)}(t_j, s) B_{q(t_j)}(s) u_{q(t_j)}(s) ds. \tag{9}$$

For $t \in (t_{j-1}, t_j]$, $j = 1, 2, \ldots, M+1$, letting

$$u_{q(t_j)}(t) = B_{q(t_j)}^T(t) X_{q(t_j)}^T(t_j, t) E_j^T \alpha \tag{10}$$

where $\alpha \in R^n$ is some constant vector to be determined.

For any given vectors $x_0, x_f \in R^n$, setting: $\beta = x_f - E_0 x_0$, then, (9) can be rewritten as

$$\tilde{\Phi}\alpha = \beta. \tag{11}$$

Since $\tilde{\Phi}$ has full rank, there exists a solution α satisfying (11). Therefore, system (1) is controllable on $[t_0, t_f]$ with respect to the switching time sequence.

Necessity: Suppose that $rank(\Phi) < n$, then, $rank(\tilde{\Phi}) < n$ and there exists a vector $x_\alpha \neq 0$ such that $\tilde{\Phi} x_\alpha = 0$. It yields that

$$x_\alpha^T \tilde{\Phi} x_\alpha = 0. \tag{12}$$

It follows from the fact that every matrix $E_j X_{q(t_j)}(t_j, s) \cdot B_{q(t_j)}(s) B_{q(t_j)}^T(s) X_{q(t_j)}^T(t_j, s) E_j^T$ is nonnegative definite and continuous on $[t_{j-1}, t_j]$, that for all $s \in [t_{j-1}, t_j]$

$$x_\alpha^T E_j X_{q(t_j)}(t_j, s) B_{q(t_j)}(s) = 0. \tag{13}$$

On the other hand, since matrix E_0 is a non-singular matrix. Thus E_0^{-1} exists. Let $x_f = 0$ and $x_0 = E_0^{-1} x_\alpha$, then, by the controllability assumption and (9), we obtain

$$\begin{aligned} 0 = x_f &= E_0 x_0 \\ &\quad + \sum_{j=1}^{M+1} E_j \int_{t_{j-1}}^{t_j} X_{q(t_j)}(t_j, s) B_{q(t_j)}(s) u_{q(t_j)}(s) ds \\ &= x_\alpha + \sum_{j=1}^{M+1} E_j \int_{t_{j-1}}^{t_j} X_{q(t_j)}(t_j, s) B_{q(t_j)}(s) u_{q(t_j)}(s) ds. \end{aligned} \tag{14}$$

By (13) and (14), we get

$$x_\alpha^T x_\alpha = -\sum_{j=1}^{M+1} \int_{t_{j-1}}^{t_j} x_\alpha^T E_j X_{q(t_j)}(t_j, s) B_{q(t_j)}(s) \cdot u_{q(t_j)}(s) ds = 0. \tag{15}$$

This is a contradiction with $x_\alpha \neq 0$ and therefore we conclude that matrix $\tilde{\Phi}$ is non-singular and $rank(\Phi) = n$. ■

Corollary 3.1: Let $D_{q(t)}(t) = 0$, $C_{q(t_0)}(t_0) = 0$ and assume that every matrix $I + C_{q(t_j)}(t_j)$ is non-singular and suppose that $A_{q(t)}(t) \equiv A_{q(t)}$, $B_{q(t)}(t) \equiv B_{q(t)}$, where $A_{q(t)}$, $B_{q(t)}$ are constant matrices. For the switching time sequence $\Sigma = \{t_i\}_{i=1}^{M+1}$, and $M \geq [n/m]$, where $[a]$ denotes the biggest integer less than or equal to a, system (1) is controllable on $[t_0, t_f]$ with respect to Σ if and only if there exists an admissible switching control input $q(t)$ such that

$$rank(\Omega) = n \tag{16}$$

$$\begin{aligned} x(t) = X_{q(t_{k+1})}^{-1}(t_{k+1}, t) \Bigg\{ &\prod_{i=k+1}^{1} X_{q(t_i)}(t_i, t_{i-1}) \Big(I + C_{q(t_{i-1})}(t_{i-1})\Big) x_0 \\ &+ \sum_{j=1}^{k} \prod_{i=k+1}^{j+1} X_{q(t_i)}(t_i, t_{i-1})(I + C_{q(t_{i-1})}(t_{i-1})) \Big) \int_{t_{j-1}}^{t_j} X_{q(t_j)}(t_j, s) B_{q(t_j)}(s) u_{q(t_j)}(s) ds \\ &+ \int_{t_k}^{t} X_{q(t_{k+1})}(t_{k+1}, s) B_{q(t_{k+1})}(s) u_{q(t_{k+1})}(s) ds \Bigg\}. \end{aligned} \tag{4}$$

where the $n \times n$ matrix Ω satisfies

$$\Omega = \left(E_1\tilde{C}_1, \ldots, E_M\tilde{C}_M, \tilde{C}_{M+1}\right) \tag{17}$$

where

$$E_j = e^{A_{q(t_{M+1})}\delta_{M+1}}(I + C_{q(t_M)})e^{A_{q(t_M)}\delta_M}(I + C_{q(t_{M-1})}) \cdots e^{A_{q(t_{j+1})}\delta_{j+1}}(I + C_{q(t_j)})$$

$\delta_i = t_i - t_{i-1}$, and $\tilde{C}_j$ is the controllable matrix of matrices $A_{q(t_j)}$ and $B_{q(t_j)}$

$$\tilde{C}_j = \left(B_{q(t_j)}, A_{q(t_j)}B_{q(t_j)}, A^2_{q(t_j)}B_{q(t_j)}, \ldots, A^{n-1}_{q(t_j)}B_{q(t_j)}\right).$$

Proof: Obviously, when $A_{q(t)}(t) \equiv A_{q(t)}, B_{q(t)}(t) \equiv B_{q(t)}$, from (6), one can obtain that

$$E_j = e^{A_{q(t_{M+1})}\delta_{M+1}}(I + C_{q(t_M)})e^{A_{q(t_M)}\delta_M}(I + C_{q(t_{M-1})}) \cdots e^{A_{q(t_{j+1})}\delta_{j+1}}(I + C_{q(t_j)}). \tag{18}$$

By Lemma 2.3, there exist functions $f_{q(t_j)_0}(t), f_{q(t_j)_1}(t), \ldots, f_{q(t_j)_{n-1}}(t)$, which are linearly independent on $[t_0, t_f]$, and hence, $f_{q(t_j)_k}(t) \neq 0, (k = 0, 1, \ldots, n-1, t \in [t_{j-1}, t_j])$, such that $X_{q(t_j)}(t_j, s) = e^{A_{q(t_j)}(t_j - s)} = \sum_{k=0}^{n-1} f_{q(t_j)_k}(t_j - s)A^k_{q(t_j)}$. Thus, we get (19) at the bottom of the page. where $\Omega_j = E_j(B_{q(t_j)} \quad A_{q(t_j)}B_{q(t_j)} \quad \cdots \quad A^{n-1}_{q(t_j)}B_{q(t_j)})$, $\theta_j(s) = \begin{pmatrix} f_{q(t_j)_0}(t_j - s)I_m \\ \vdots \\ f_{q(t_j)_{n-1}}(t_j - s)I_m \end{pmatrix}$, I_m is the $m \times m$ identify matrix, and for $j = 1, 2, \ldots, M+1$, $\Theta_j = \int_{t_{j-1}}^{t_j} \theta_j(s)\theta_j^T(s)ds$.

We have that

$$\tilde{\Phi} = \sum_{j=1}^{M+1} \Phi_j = \sum_{j=1}^{M+1} \Omega_j\Theta_j\Omega_j^T = \Omega\Theta\Omega^T \tag{20}$$

where $\Omega = (\Omega_1 \ \Omega_2 \ \cdots \ \Omega_{M+1}) \in R^{n \times nm(M+1)}$, $\Theta = \text{diag}\left\{\Theta_1, \Theta_2, \ldots, \Theta_{M+1}\right\} \in R^{nmM \times nm(M+1)}$.

Thus, by (20) and Theorem 3.1, the system is controllable if and only if

$$rank(\tilde{\Phi}) = rank(\Omega\Theta\Omega^T) = n. \tag{21}$$

To complete the proof, there remains to show that

$$rank(\tilde{\Phi}) = rank(\Omega). \tag{22}$$

From the fact that for any $t \in (t_{j-1}, t_j]$, $f_{q(t_j)_i}(t), i = 0, 1, \ldots, n-1$, is linearly independent, we get that $rank(\Theta_j) = m$ and hence $rank(\Theta) = m(M+1)$. Moreover, from $M \geq [n/m]$, we get $rank(\Theta) = m(M+1) \geq n$. Thus, from the fact that Θ is a nonnegative definite symmetric matrix with $rank(\Theta) \geq n$, we obtain $rank(\Omega) = rank(\Omega\Theta\Omega^T) = rank(\tilde{\Phi})$. Hence, (22) holds. Thus, by Theorem 3.1, the corollary is true. ∎

Corollary 3.2: Assume that all conditions of Corollary 3.1 hold, then the sufficient and necessary condition for system (1) to be controllable with respect to Σ is that there exists an admissible switching control input $q(t)$ such that matrix $\tilde{\Omega}$ has full rank, where

$$\tilde{\Omega} = \left(\tilde{C}_1, \ldots, \tilde{C}_M, \tilde{C}_{M+1}\right) \tag{23}$$

where $\tilde{C}_j$ is the controllable matrix of matrices $A_{q(t_j)}$ and $B_{q(t_j)}$ defined as in Corollary 3.1.

Proof: Since $I + C_{q(t_j)}(t_j)$ is non-singular, the matrix E_j is invertible. Moreover, it is easy to see that

$$rank\{\Omega\} \leq rank\{\tilde{\Omega}\} \leq n. \tag{24}$$

Thus, $rank(\Omega) = rank(\tilde{\Omega}) = n$. By Corollary 3.1, the result of this corollary is true. ∎

Remark 3.2: If system (1) is time-invariant (i.e., $A_i(t) = A_i, B_i(t) = B_i, (i = 1, 2, \ldots, r)$) and no impulses happened in it (i.e., $C_i(t) = 0, D_i(t) = 0, (i = 1, 2, \ldots, r)$), results in Corollaries 3.1–3.2 include those results for switching systems in the literature, (see e.g., [11], [15], [18], [20]).

B. *Only the Discrete Control Signals are Input:* $B_{q(t)}(t) = 0$

For convenience, let $D_{q(t_0)}(t_0) = 0$ and denote

$$W_j = \prod_{i=M+1}^{j+1} X_{q(t_i)}(t_i, t_{i-1}), \quad j = 0, 1, \ldots, M. \tag{25}$$

Then, it follows from Lemma 2.1 and (3) and (25) that

$$x(t_f) = W_0x_0 + \sum_{j=1}^{M} W_jC_{q(t_j)}(t_j)x_{q(t_j)}(t_j) + \sum_{j=1}^{M} W_jD_{q(t_j)}(t_j)v_{q(t_j)}(t_j). \tag{26}$$

Theorem 3.2: The switching impulsive system (1) with $B_{q(t)}(t) = 0, C_{q(t_0)}(t_0) = 0$ and $D_{q(t_0)}(t_0) = 0$ is controllable on $[t_0, t_f]$ with respect to the switching time sequence $\Sigma = \{t_i\}_{i=1}^{M+1}$ if and only if there exists an admissible switching control input $q(t)$ such that

$$\Psi = \left(W_1D_{q(t_1)}(t_1), W_2D_{q(t_2)}(t_2), \ldots, W_MD_{q(t_M)}(t_M)\right) \tag{27}$$

has full rank.

Proof: By Definition 3.1 and Lemma 2.1, for the given $\Sigma = \{t_i\}_{i=1}^{M+1}$ with $t_{M+1} = t_f$, and every $x_0, x_f \in R^n$, system (1) is controllable on $[t_0, t_f]$ if and only if there exist an admissible switching control input $q(t)$ and discrete control inputs $v_{q(t)}(t)$ such that

$$x_f = x(t_f) = W_0x_0 + \sum_{j=1}^{M} W_jC_{q(t_j)}(t_j)x_{q(t_j)}(t_j) + \sum_{j=1}^{M} W_jD_{q(t_j)}(t_j)v_{q(t_j)}(t_j). \tag{28}$$

$$\begin{aligned}\Phi_j &= \int_{t_{j-1}}^{t_j} E_jX_{q(t_j)}(t_j, s)B_{q(t_j)}(s)B^T_{q(t_j)}(s) \cdot X^T_{q(t_j)}(t_j, s)E_j^T ds \\ &= \sum_{i=0}^{n-1}\sum_{k=0}^{n-1} \int_{t_{j-1}}^{t_j} f_{q(t_j)_i}(t_j - s)f_{q(t_j)_k}(t_j - s)ds \cdot E_jA^i_{q(t_j)}B_{q(t_j)}B^T_{q(t_j)}(A^i_{q(t_j)})^TE_j^T \\ &= \Omega_j \int_{t_{j-1}}^{t_j} \theta_j(s)\theta_j(s)^T ds\Omega_j^T = \Omega_j\Theta_j\Omega_j^T\end{aligned} \tag{19}$$

Denote $Y = (v^T_{q(t_1)}(t_1), v^T_{q(t_2)}(t_2), \ldots, v^T_{q(t_M)}(t_M))^T$ and

$$\beta = x_f - W_0 x_0 - \sum_{j=1}^{M} W_j C_{q(t_j)}(t_j) x_{q(t_j)}(t_j).$$

Then, it follows from (28) that

$$\Psi Y = \beta, \tag{29}$$

which implies that $\beta \in \mathrm{Span}\{\Psi\}$. Thus, for any $x_0, x_f \in R^n$,(29) has a solution Y if and only if $\mathrm{Span}\{\Psi\} = R^n$. That is, if and only if matrix Ψ has full rank. ∎

Corollary 3.3: For the $\Sigma = \{t_i\}_{i=1}^{M+1}$ with $t_{M+1} = t_f$, denote $\delta_i = t_i - t_{i-1}$. Assume that $A_{q(t)}(t) \equiv A_{q(t)}$, $D_{q(t)}(t) \equiv D_{q(t)}$, where $A_{q(t)}, D_{q(t)}$ are constant matrices, then the switching impulsive system (1) with $B_{q(t)}(t) = 0, C_{q(t_0)}(t_0) = 0$ and $D_{q(t_0)}(t_0) = 0$ is controllable on $[t_0, t_f]$ with respect to Σ if and only if there exists a switching input $q(t)$ such that

$$\begin{aligned}\Psi = \Big(& e^{A_{q(t_{M+1})}\delta_{M+1}} e^{A_{q(t_M)}\delta_M} \cdots e^{A_{q(t_2)}\delta_2} D_{q(t_1)}, \\ & e^{A_{q(t_{M+1})}\delta_{M+1}} e^{A_{q(t_M)}\delta_M} \cdots e^{A_{q(t_3)}\delta_3} D_{q(t_2)}, \\ & \ldots, e^{A_{q(t_{M+1})}\delta_{M+1}} D_{q(t_M)}\Big)\end{aligned} \tag{30}$$

has full rank.

Proof: If $A_{q(t)}(t) \equiv A_{q(t)}$, where $A_{q(t)}$ are constant matrices, then

$$X_{q(t_i)}(t_i, t_{i-1}) = e^{A_{q(t_i)}\delta_i}, i = 1, 2, \ldots, M. \tag{31}$$

The result is the direct consequence of Theorem 3.2. ∎

Corollary 3.4: For the $\Sigma = \{t_i\}_{i=1}^{M+1}$ with $t_{M+1} = t_f$, denote $\delta_i = t_i - t_{i-1}$. Assume that $A_{q(t)}(t) \equiv A, D_{q(t)}(t) \equiv D$, where A, D are constant matrices, then system (1) is controllable on $[t_0, t_f]$ with respect to Σ if and only if

$$rank(D, AD, A^2D, \ldots, A^{n-1}D) = n. \tag{32}$$

Proof: When $A_{q(t)}(t) \equiv A, D_{q(t)}(t) \equiv D$, it follows that $W_j = e^{A\gamma_j}$, where $\gamma_j = t_{M+1} - t_j, j = 1, 2, \ldots, M$. By Lemma 2.3, there exist linearly independent scalar functions $f_i(t), i = 0, 1, \ldots, n-1$, such that $e^{At} = \sum_{i=0}^{n-1} f_i(t)A^i$. Thus, we get

$$W_j D = e^{A\gamma_j} D = \sum_{i=0}^{n-1} f_i(\gamma_j) A^i D \tag{33}$$

which implies that $W_j D \in \mathrm{Span}\{D, AD, A^2D, \ldots, A^{n-1}D\}, j = 1, 2, \ldots, M$.

Hence, $\mathrm{Span}\{\Psi\} \subseteq \mathrm{Span}\{D, AD, A^2D, \ldots, A^{n-1}D\}$. That is

$$rank(\Psi) \le rank(D, AD, A^2D, \ldots, A^{n-1}D) \le n. \tag{34}$$

On other hand, by Theorem 3.2, system (1) is controllable with respect to Σ if and only if the matrix Ψ has full rank. Therefore, by (34), system (1) is controllable with respect to Σ if and only if (32) holds. ∎

Remark 3.3: If system (1) is time-invariant (i.e., $A_i(t) = A_i, B_i(t) = 0, C_i(t) = C_i, D_i(t) = D_i, (i = 1, 2, \ldots, r)$), results in Corollaries 3.3–3.4 include those results for impulsive hybrid systems in the literature, (see e.g., [12], [21]).

IV. Controlled Observability

In this section, we first define the concept of *controlled observability* for time-varying switching impulsive systems. Then we establish the criteria of controlled observability for this class of systems.

Definition 4.1: The switching impulsive systems (1) is called to be controlled observable on $[t_0, t_f]$ $(t_f > t_0)$ with respect to the switching time sequence $\Sigma = \{t_i\}_{i=1}^{M+1}$ with $t_{M+1} = t_f$ if there exists an admissible switching control input $q(t)$ such that any initial state $x_0 \in R^n$ is uniquely determined by the corresponding system inputs $u_{q(t)}(t)$ (or $v_{q(t)}(t)$) and system output $y(t)$ for $t \in [t_0, t_f]$.

Theorem 4.1: The switching impulsive system (1) with $D_{q(t)}(t) = 0, C_{q(t_0)}(t_0) = 0$ is controlled observable with respect to the $\Sigma = \{t_i\}_{i=1}^{M+1}$ with $t_{M+1} = t_f$ if and only if there exists an admissible switching control input $q(t)$ such that the $n \times n$ matrix Γ satisfies $rank(\Gamma) = n$, where

$$\Gamma = (\Gamma_1,\ \Gamma_2,\ \cdots,\ \Gamma_{M+1})$$

and for $k = 0, 1, \ldots, M,\ t \in (t_k, t_{k+1}]$,

$$\begin{aligned}&\Gamma_{k+1} \\ &= \int_{t_k}^{t_{k+1}} \Gamma^T_{k+1}(s, t_0) F^T_{q(t_{k+1})}(s) \cdot F_{q(t_{k+1})}(s) \Gamma_{k+1}(s, t_0) ds;\end{aligned} \tag{35}$$

$$\begin{aligned}&\Gamma_{k+1}(t, t_0) \\ &= X^{-1}_{q(t_{k+1})}(t_{k+1}, t) \prod_{i=k+1}^{1} X_{q(t_i)}(t_i, t_{i-1}) \cdot \Big(I + C_{q(t_{i-1})}(t_{i-1})\Big).\end{aligned} \tag{36}$$

Proof:

Sufficiency: Letting the control inputs $u_{q(t)}(t) = 0$, it follows from Lemma 2.2 that for any $t \in (t_k, t_{k+1}], k = 0, 1, \ldots, M$,

$$y(t) = F_{q(t_{k+1})}(t) x(t) = F_{q(t_{k+1})}(t) \Gamma_{k+1}(t, t_0) x_0. \tag{37}$$

Multiplying (37) by $\Gamma_{k+1}(t, t_0)^T F_{q(t_{k+1})}(t)^T$ and integrating from t_0 to t_f, we have

$$\begin{aligned}&\sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} \Gamma_{k+1}(s, t_0)^T F_{q(t_{k+1})}(s)^T y(s) ds \\ &\qquad = \sum_{k=0}^{M} \Gamma_{k+1} x_0 = \tilde{\Gamma} x_0\end{aligned} \tag{38}$$

where $\tilde{\Gamma} = \sum_{k=0}^{M} \Gamma_{k+1}$.

Thus, if $rank(\Gamma) = n$, then $rank(\tilde{\Gamma}) = n$ and hence it follows that:

$$x_0 = \tilde{\Gamma}^{-1} \sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} \Gamma_{k+1}(s, t_0)^T F_{q(t_{k+1})}(s)^T y(s) ds. \tag{39}$$

Hence, x_0 is uniquely determined by $y(t)$ and therefore the system is controlled observable.

Necessity: Suppose $rank(\Gamma) < n$, then, the symmetry matrix $\tilde{\Gamma}$ is singular and there exists a nonzero vector $\alpha \in R^n$ such that

$$\alpha^T \tilde{\Gamma} \alpha = 0. \tag{40}$$

Since every matrix $\Gamma^T_k(t, t_0) F^T_{q(t_k)}(t) F_{q(t_k)}(t) \Gamma_k(t, t_0)$ is nonnegative definite and its entries all are continuous, for $t \in (t_{k-1}, t_k], k = 1, 2, \ldots, M+1$, we get

$$F_{q(t_k)}(t) \Gamma_k(t, t_0) \alpha = 0. \tag{41}$$

For $t \in (t_k, t_{k+1}]$, define $\tilde{y}(t)$ as: See equation (42) at the bottom of the next page. From Definition 4.1, it is easy to see that system (1) is controlled observable is equivalent to x_0 being uniquely determined by $\tilde{y}(t)$. Let $x_0 = \alpha$, by Lemma 2.2 and (41)–(42), we have

$$\begin{aligned}\tilde{y}(t) &= F_{q(t_{k+1})}(t)X^{-1}_{q(t_{k+1})}(t_{k+1}, t) \\ &\quad \cdot \prod_{i=k+1}^{1} X_{q(t_i)}(t_i, t_{i-1}) \cdot \Big(I + C_{q(t_{i-1})}(t_{i-1})\Big)x_0 \\ &= F_{q(t_{k+1})}(t)\Gamma_{k+1}(t, t_0)x_0 = 0. \end{aligned} \tag{43}$$

Thus, for $t \in (t_k, t_{k+1}]$, $k = 0, 1, \ldots, M$, we obtain

$$0 = \tilde{y}(t) = F_{q(t_{k+1})}(t)\Gamma_{k+1}(t, t_0)x_0 \tag{44}$$

which yields that system (1) is not controlled observable on $[t_0, t_f]$. This is a contradiction and therefore the matrix Γ satisfies $rank(\Gamma) = n$. ■

Theorem 4.2: The switching impulsive system (1) with $B_{q(t)}(t) = 0, C_{q(t_0)}(t_0) = 0$, is controlled observable with respect to $\Sigma = \{t_i\}_{i=1}^{M+1}$ with $t_{M+1} = t_f$ if and only if there exists an admissible switching control input $q(t)$ such that the $n \times n$ matrix

$$\Upsilon = \sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} \Upsilon^T_{k+1}(s, t_0)F^T_{q(t_{k+1})}(s) \cdot F_{q(t_{k+1})}(s)\Upsilon_{k+1}(s, t_0)ds \tag{45}$$

is non-singular, where for $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, M$

$$\Upsilon_{k+1}(t, t_0) = X^{-1}_{q(t_{k+1})}(t_{k+1}, t) \prod_{i=k+1}^{1} X_{q(t_i)}(t_i, t_{i-1}). \tag{46}$$

Proof: The proof follows an argument analogous to that of Theorem 4.1 and is omitted. ■

Corollary 4.1: For the given $\Sigma = \{t_i\}_{i=1}^{M+1}$ with $t_{M+1} = t_f$, denote $\delta_i = t_i - t_{i-1}$. Assume that $A_{q(t)}(t) \equiv A_{q(t)}$, $F_{q(t)}(t) \equiv F_{q(t)}$, where $A_{q(t)}, F_{q(t)}$ are constant matrices, and furthermore suppose that every matrix $I + C_{q(t_j)}(t_j)$ is non-singular. Then, system (1) with $B_{q(t)}(t) = 0, C_{q(t_0)}(t_0) = 0$, (or with $D_{q(t)}(t) = 0, C_{q(t_0)}(t_0) = 0$,) is controlled observable on $[t_0, t_f]$ with respect to $\Sigma = \{t_i\}_{i=1}^{M+1}$ if and only if that there exists an admissible switching control input $q(t)$ such that the matrix $S = \Big(S_{q(t_1)}, S_{q(t_2)}, \ldots, S_{q(t_{M+1})}\Big)$ satisfies $rank(S) = n$, where $S_{q(t_j)}$ is the observable matrix of matrices $A_{q(t_j)}$ and $F_{q(t_j)}$

$$S_{q(t_j)} = \begin{pmatrix} F_{q(t_j)} \\ F_{q(t_j)}A_{q(t_j)} \\ F_{q(t_j)}A^2_{q(t_j)} \\ \vdots \\ F_{q(t_j)}A^{n-1}_{q(t_j)} \end{pmatrix}, \quad j = 1, 2, \ldots, M + 1. \tag{47}$$

Proof: The proof is similar to that for Corollary 3.1. The details are omitted. ■

Remark 4.1:

1) In [29], the observer-based stabilization problem for switching single input and single output linear systems is studied. The results derived in this note are less conservative than those in [29]. Hence, by using the results obtained in this note and the methods in [29], for the given switching time sequence, we can study the design of observer-based controllers for the stabilization of time-variant switching impulsive systems. This will be a topic of further research.
2) The results obtained in Theorems 3.1–3.2 and Theorems 4.1–4.2 can be used to design the switching modes for switching systems to achieve the controllability and controlled observability, respectively.

V. Example

In this section, we give an example to illustrate the obtained results.

Example 5.1: Consider the time-variant system with two subsystems in form of (1). Here, the matrices are given as: $C_1(0) = 0, C_2(0) = 0$, and $A_1(t) = \begin{pmatrix} -1 & -e^{2t} \\ 0 & -4 \end{pmatrix}$, $B_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $C_1(t) = -0.2I$, $D_1(t) = 0$, $F_1(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; $A_2(t) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$, $B_2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C_2(t) = 0.2I$, $D_2(t) = 0$, $F_2(t) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

For the given switching time sequence $\Sigma = \{t_i\}_{i=0}^{M+1}$, where $M = 2, t_0 = 0, t_1 = 1, t_2 = 2, t_3 = t_f = 2.5$, we choose an admissible switching control as

$$\begin{aligned} q(t) &= q(t_0) = q(t_1) = 1, \quad \text{for} \quad t \in (t_0, t_1], \\ q(t) &= q(t_1^+) = q(t_2) = 2, \quad \text{for} \quad t \in (t_1, t_2], \\ q(t) &= q(t_2^+) = q(t_f) = 1, \quad \text{for} \quad t \in (t_2, t_f]. \end{aligned}$$

By calculation, we get the transition matrices associated $A_1(t)$ and $A_2(t)$, respectively

$$\begin{aligned} X_1(t, s) &= \begin{pmatrix} e^{-t+s} & -e^{-t+3s} + e^{-2t+4s} \\ 0 & e^{-4t+4s} \end{pmatrix} \\ X_2(t, s) &= \begin{pmatrix} e^{t-s} & 0 \\ 0 & e^{-2(t-s)} \end{pmatrix}. \end{aligned}$$

Thus, we further get that

$$\begin{aligned} \Phi_1 &= \begin{pmatrix} 5.7171 & -0.0129 \\ -0.0129 & 0.0000 \end{pmatrix}, \Phi_2 = \begin{pmatrix} 1.6922 & 0 \\ 0 & 0 \end{pmatrix}, \\ \Phi_3 &= \begin{pmatrix} 327.0740 & -4.4064 \\ -4.4064 & 0.1206 \end{pmatrix}. \end{aligned}$$

It follows from $rank(\Phi_1, \Phi_2, \Phi_3) = 2$ and Theorem 3.1 that the system is controllable on $[t_0, t_f]$ with respect to the switching time sequence.

In fact, let $x_0 = 0$, and for some constants $c_1, c_2 \in R$, let $u_{q(t)}(t) = \begin{cases} u_1(t) = c_1, & \text{if } q(t) = 1, \\ u_2(t) = c_2, & \text{if } q(t) = 2, \end{cases}$ then, by Lemma 2.2, we obtain that $x(t_f) = x(2.5) = \begin{pmatrix} 17.6233 \\ -0.2205 \end{pmatrix} c_1 + \begin{pmatrix} 1.2506 \\ 0 \end{pmatrix} c_2$.

$$\begin{aligned} \tilde{y}(t) &= y(t) - F_{q(t_{k+1})}(t)X^{-1}_{q(t_{k+1})}(t_{k+1}, t) \\ &\quad \cdot \Bigg\{ \sum_{j=1}^{k} \prod_{i=k+1}^{j+1} X_{q(t_i)}(t_i, t_{i-1})(I + C_{q(t_{i-1})}(t_{i-1})\Big) \cdot \int_{t_{j-1}}^{t_j} X_{q(t_j)}(t_j, s)B_{q(t_j)}(s)u_{q(t_j)}(s)ds \\ &\quad + \int_{t_k}^{t} X_{q(t_{k+1})}(t_{k+1}, s)B_{q(t_{k+1})}(s)u_{q(t_{k+1})}(s)ds \Bigg\}. \end{aligned} \tag{42}$$

Obviously, $x(t_f)$ can be driven anywhere by selecting proper c_1 and c_2. Thus, the system is controllable on $[t_0, t_f]$ with respect to the switching time sequence.

Moreover, by calculation, we get that

$$\Gamma_1 = \begin{pmatrix} 0.4323 & 0.0831 \\ 0.0831 & 0.2294 \end{pmatrix}, \Gamma_2 = \begin{pmatrix} 0.2767 & -0.1776 \\ -0.1776 & 0.0036 \end{pmatrix},$$
$$\Gamma_3 = \begin{pmatrix} 0.2913 & -0.1908 \\ -0.1908 & 0.1251 \end{pmatrix}.$$

Thus, $rank(\Gamma_1, \Gamma_2, \Gamma_3) = 2$ and hence, by Theorem 4.1, we obtain that the system is controlled observable. In fact, let $u_1(t) = u_2(t) = 0$, we get that

$$y(t) = \begin{pmatrix} e^{-t} & -e^{-t} + e^{-2t} + e^{-4t} \\ 0 & e^{-4t} \end{pmatrix} x_0, \ t \in (t_0, t_1],$$
$$y(t) = 0.8e^{t-2} \begin{pmatrix} 1 & -1 + e^{-1} - e^{-3t} \\ 0 & e^{-3t} \end{pmatrix} x_0, \ t \in (t_1, t_2]$$

and for $t \in (t_2, t_f]$,

$$y(t) = 0.96e^{-t+2} \begin{pmatrix} 1 & -e^{-2} + e^{-1} - 1 + e^{-t} - e^{-3t} \\ 0 & e^{-3t} \end{pmatrix} x_0.$$

It yields that $x_0 \in R^n$ is uniquely determined by $y(t)$ for any $t \in [t_0, t_f]$.

VI. Conclusion

In this note, we have studied the controllability and controlled observability problems for a class of time-varying controlled switching impulsive systems. Necessary and sufficient conditions for controllability and controlled observability were derived and expressed in the form of matrix rank conditions that can be easily tested. Restricting attention to time-invariant linear switching impulsive systems, corresponding algebraic criteria were also derived. These criteria include those results for classical linear systems and time-variant linear switching systems previously reported in the literature.

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